4. Complex Number

☆ Power of iota (i)

$i = i$ $\qquad$ $i^5 = i$

$i^2 = -1$ $\qquad$ $i^6 = -1$

$i^3 = -i$ $\qquad$ $i^7 = -i$

$i^4 = 1$ $\qquad$ $i^8 = 1$

Note : $i + i^2 + i^3 + i^4 = 0$

$(i^{4k}) = 1$

$\frac{1}{i} = -i$

Sum of four iota's whose power is conjucative them its value is 0

☆ Standard form of Complex number

$z = a + ib$

Complex number denoted by z

$a =$ Real part of $z = Re(z)$

$b =$ Imaginary part of $z = Im(z)$

Purely real number $= b = 0$

Purely imaginary number $= a = 0$

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Operations on Complex number

1 Equality of complex number

If two Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal

$$z_1 = z_2$$

$$a_1 + ib_1 = a_2 + ib_2$$

$$(a_1 - a_2) + (ib_1 - ib_2) = 0$$

$$(a_1 - a_2) + i(b_1 - b_2) = 0$$

$$a_1 - a_2 = 0 \qquad b_1 - b_2 = 0$$

$$a_1 = a_2 \qquad b_1 = b_2$$

2. Addition / Subtraction of Complex number

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2)$$

$$(a_1 - a_2) + (b_1 - b_2)$$

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Multiplication of Complex number

$$z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2)$$

$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

Division of Complex number

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2)^2 - (ib_2)^2}$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2)^2 + (b_2)^2}$$

Additive and Multiplicative Inverse

Additive identity $= 0$

Multiplicative identity $= 1$

Additive inverse

$x + b =$ Additive identity
$= 0$

b is additive inverse of x
x is additive inverse of b

Multiplicative inverse

Complex number z_1 and z_2 is said to be Multiplicative inverse of eath other

$$z_1 \cdot z_2 = 1$$

$$\boxed{z_1 = \frac{1}{z_2}}$$

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★ Square root of complex ~~mu~~ number

$z = x + iy$ we want to find
$\sqrt{z} = \sqrt{x+iy}$

$$\sqrt{x+iy} = a+ib$$

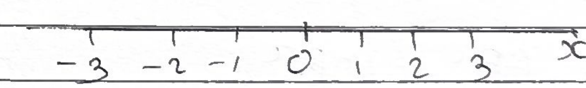
$$x+iy = (a+ib)^2$$

$$x+iy = a^2-b^2+(2ab)i$$

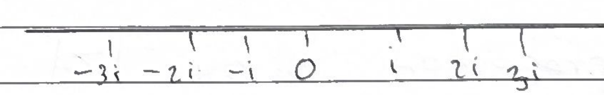
$$x = a^2-b^2 \qquad y = 2ab$$

★ Representation of Complex number

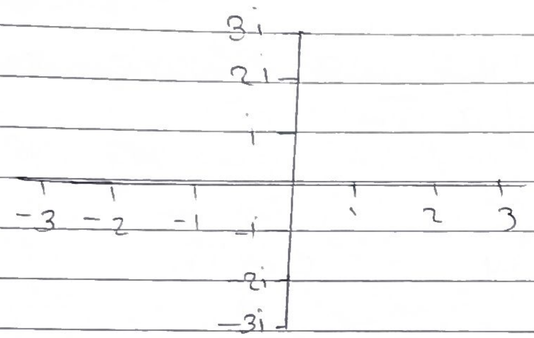
Real line



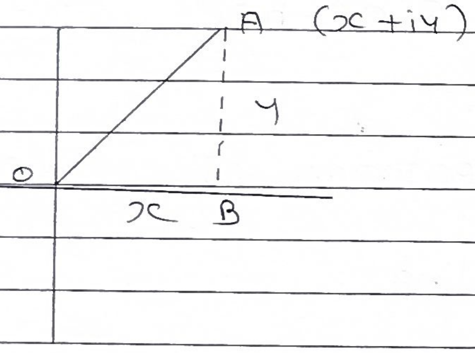
Imaginary line



Argand plane



Modulus : Distance from origin



$OA^2 = AB^2 + OB^2$

$OA^2 = y^2 + x^2$

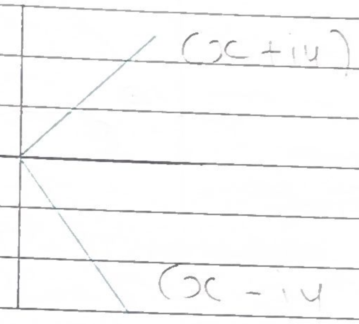
$|z| = \sqrt{x^2 + y^2}$

It represented by $|z|$

Conjugate of complex number

If $z = x + iy$ the Conjugate of $\bar{z} = x - iy$

Conjugate of complex number reprent a mirror image about real axis

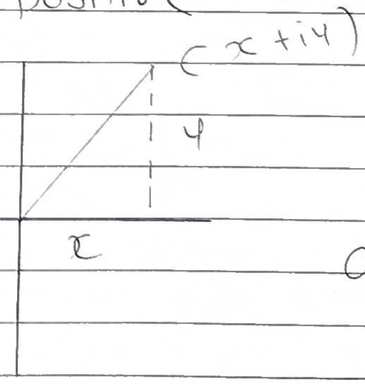


It is represented by $\bar{z}$

✱ Argument of Complex numbers

Defination = Argument

Argument is a angle of position vector Complex number with positive x - axis Real axis) positive

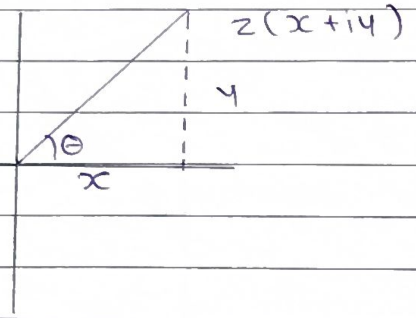


$arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Principle Argument

$-\pi < \theta \le \pi$

Argument in 1st qudrant



$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$$

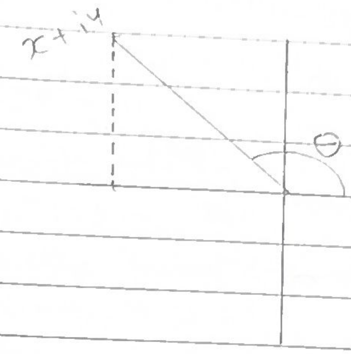
$$= \tan\left(\frac{y}{x}\right)$$

$$= \arg(z)$$

$$\alpha = \arg(z) = \tan^{-1}\left|\frac{y}{x}\right|$$

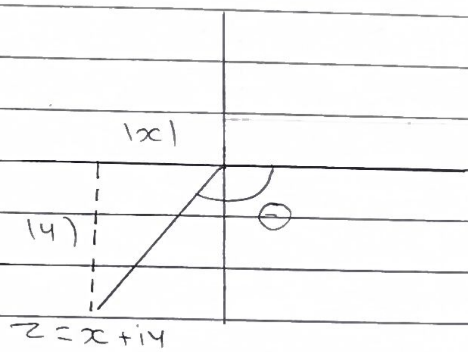
$$\boxed{\theta = \alpha}$$

Argument in 2nd quadrant



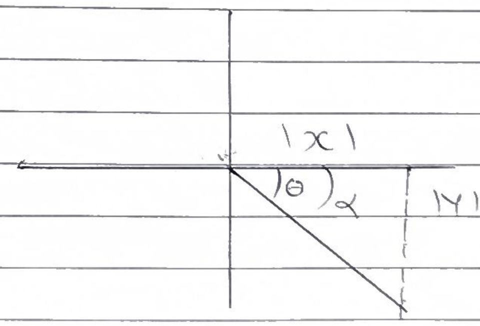
$\theta = \pi - \alpha$

Argument in 3rd quadrant



$$\boxed{\theta = -\pi + \alpha}$$

Argument in 4th quadrant

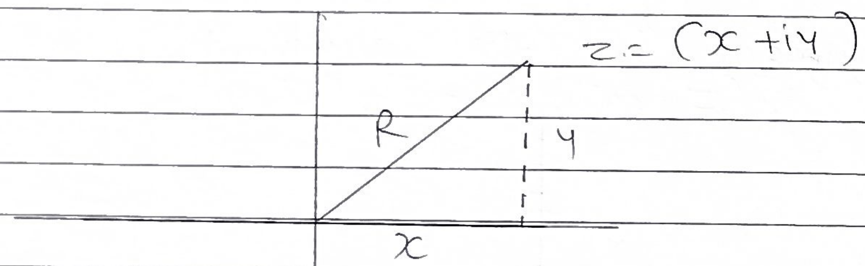


$z = x + iy$

$\theta = -\alpha$

$\theta = \pi - \alpha$	$\theta = \alpha$
$\theta = -\pi + \alpha$	$\theta = -\alpha$

Polar representation of Complex number



$z = \alpha(\cos\theta + i\sin\theta)$

$r = |z|$ r is a mod $|z|$

$-\pi < \theta \leq \pi$

Euler form of Complex number

$z = x + iy$

Euler form of $z = r.e^{i\theta}$

where $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \frac{x^6}{6!} + \frac{x^8}{8!}$$

$e^{i\pi} + 1 = 0$ is a eulers identity

☆ Value of eulers identity

$$\boxed{e^{i\pi} = -1}$$

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✲ Properties of Conjugate ($\bar{z}$)

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2Re(z)$
3. $z - \bar{z} = 2im(z)$
4. $z = \bar{z} \rightarrow z$ is purely real
5. $z + \bar{z} = 0$ z is purely imaginary
6. $z.\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2$
7. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
8) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
9) $\overline{z_1 z_2} = \overline{z_1} . \overline{z_2}$
10) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ $z_2 \neq 0$

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Propertis of Modulus

1. $|z| = |\bar{z}|$ $|-z| = |-\bar{z}|$
2. $Re(z) \leq |z|$, $Im(z) \leq |z|$
3. $Re(z) \geq -|z|$, $Im(z) \geq -|z|$
4. $|z_1 . z_2| = |z_1| . |z_2|$
5. $z . \bar{z} = |z|^2$
6) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
7. $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2Re(z_1 . \bar{z}_2)$

Propertis of Argument

i) $arg(\bar{z}) = -arg(z)$
ii) $arg(z_1 z_2) = arg(z_1) + arg(z_2)$
iii) $arg(z_1 \bar{z}_2) = arg(z_1) - arg(z_2)$
iv) $arg\left[\frac{z_1}{z_2}\right] = arg(z_1) - arg(z_2)$

✫ properties of log function

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^n = n \log_a x$
4. $\log (x \cdot y) = \log_a x + \log_a y$
5. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
6. $\log_{a^b} (m) = \frac{1}{b} \log_a (m)$
7. $\log_a m = \frac{\log_b m}{\log_b a}$
8. $\log_a m = \frac{1}{\log_m a}$
9. $a^{\log_a x} = x$
10. $a^{\log_b c} = c^{\log_b a}$
11. $\ln x = \log_e x$ $e = 2.73$ exponental

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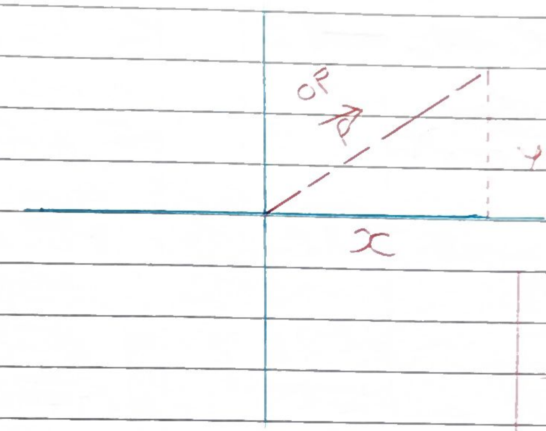
Complex number as vector

Argand Plane



$z = x + iy$

$\vec{P} = (x, y) = x\hat{i} + y\hat{j}$

$\vec{P} = x\hat{i} + y\hat{j} = \vec{OP}$

$z = x + iy$

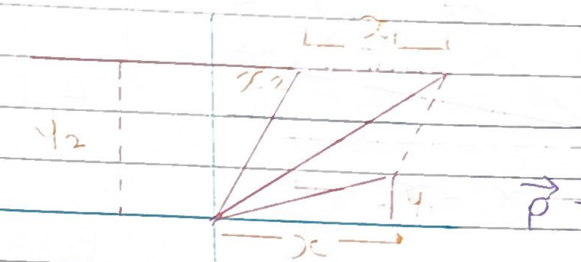
Magnitude of vector & modulus of complex number

$|\vec{OP}| = \sqrt{x^2+y^2}$

...(length of vector)

$|z| = \sqrt{x^2+y^2}$

Parellelogram law



$\vec{R} = (x_1+x_2)\hat{i} + (y_1+y_2)\hat{i}$

$\vec{P} = x_1\hat{i} * y_1\hat{i}$

$\vec{P} = x_1\hat{i} + y_1\hat{i}$

$\vec{Q} = x_2\hat{i} + y_2\hat{i}$

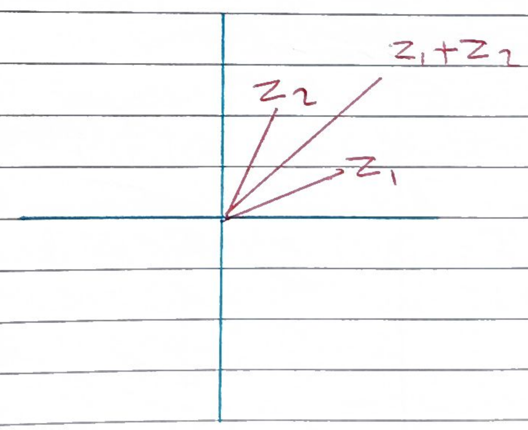
$\vec{R} = x_1\hat{i} + x_2\hat{i} + y_1\hat{i} + y_2\hat{i}$

$= x_1\hat{i} + y_1\hat{i} + x_2\hat{i} + y_2\hat{i}$

$\vec{R} = \vec{P} + \vec{Q}$

$\vec{P} = z_1 = x_1 + iy_1$

$\vec{Q} = z_2 = x_2 + iy_2$



Note

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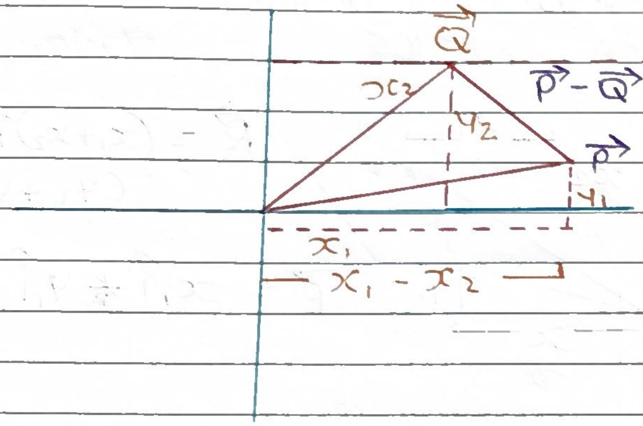

Note.

$A \longrightarrow B$

A is a Head of Vector

B is a tail of vector

$\vec{S} = \vec{P} - \vec{Q} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{i}$

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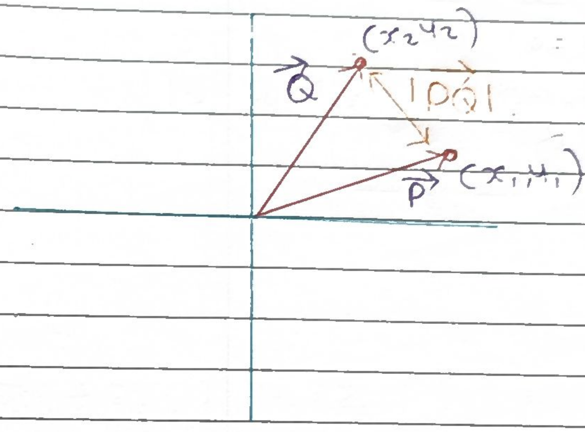
Distance between two Complex number

Note



$\because z_1 = \vec{P} = x_1\hat{i} + y_1\hat{i}$

$z_2 = \vec{Q} = x_2\hat{i} + y_2\hat{i}$

$z_1 = x_1 + iy \qquad z_2 = x_2 + iy$

Distance between Point P&Q

$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$\vec{PQ} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{i}$

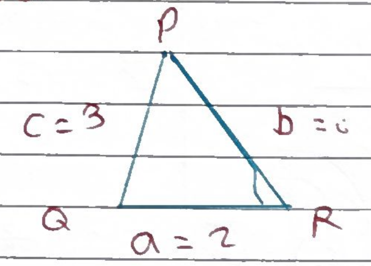
$= \vec{P} - \vec{Q}$

$|\vec{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$(z_1 - z_2) = (x_1 - x_2) + i(y_1 - y_2)$

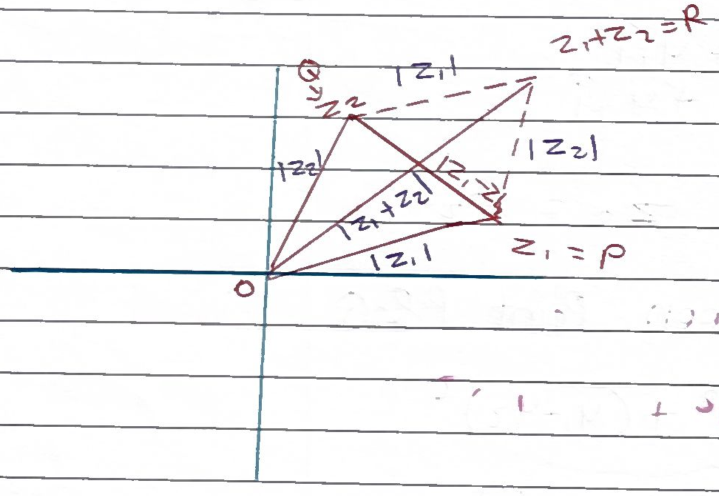
Triangle inequality for complex number

Note



$a+c>b$

$|a-c|<b$



For triangle ΔOPR

$||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$

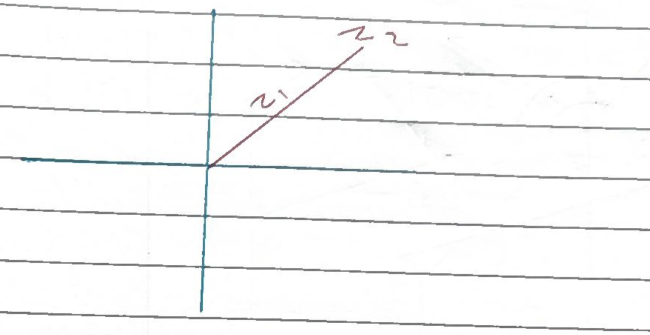
For ΔOQP

$||z_1|-|z_2|| \le |z_1-z_2| \le |z_1|+|z_2|$

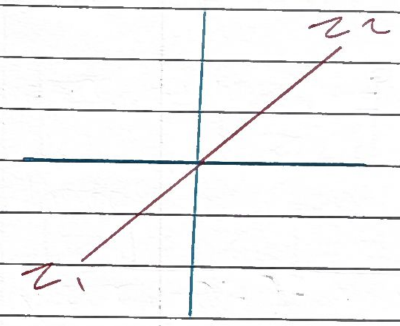
$||z_1|-|z_2|| = |z_1-z_2| = |z_1|+|z_2|$

When z_1, z_2 are colinear

Note



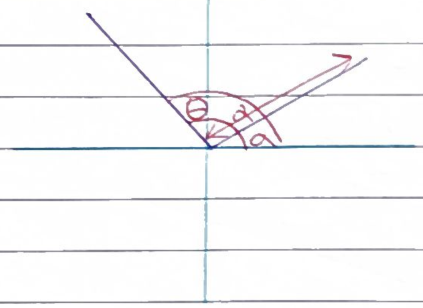
$|z_1+z_2| = |z_1|+|z_2|$



$||z_1|-|z_2|| = |z_1+z_2|$

Rotation theorem

Case I



$$z_1 = r.e^{ir}$$

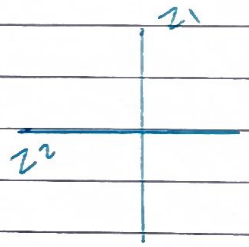
$$z_2 = r.e^{i(\theta+\alpha)}$$

$$z_2 = r.e^{i\theta}.e^{i\alpha}$$

$$z_2 = r.e^{i\alpha}.e^{i\theta}$$

$$\boxed{z_2 = z_1 e^{i\theta}}$$

$$z_1 = z_2 . e^{i\theta}$$
$$z_2 = z_1 . e^{i\theta}$$

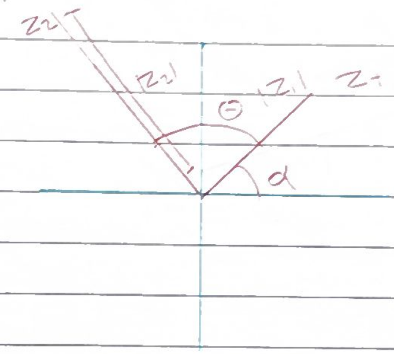


$$z_2 = z_1 e^{i\left(-\frac{\pi}{2}\right)}$$

$$z_1 = z_2 e^{i\left(\frac{\pi}{2}\right)}$$

Case II $z_1 = |z_1| . e^{i\alpha}$

$$\left|\frac{z_1}{z_1}\right| - ?$$



$$\boxed{z_3 = \frac{z_1}{|z_1|}}$$

$$\boxed{z_4 = \frac{z_2}{|z_2|}}$$

$$|z_3| = |z_4| = 1$$

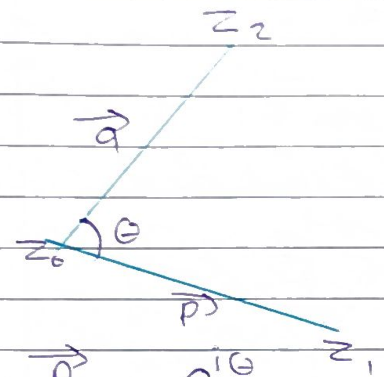
$$\boxed{|z_3| = |z_4|}$$

$$z_4 = z_3 e^{i\theta}$$

$$\frac{z_2}{|z_2|} = \frac{z_1}{|z_1|} . e^{i\theta}$$

$$z_2 = \frac{|z_2|}{|z_1|} . z e^{i\theta}$$

Case III



$$\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{p}}{|\vec{p}|} \cdot e^{i\theta}$$

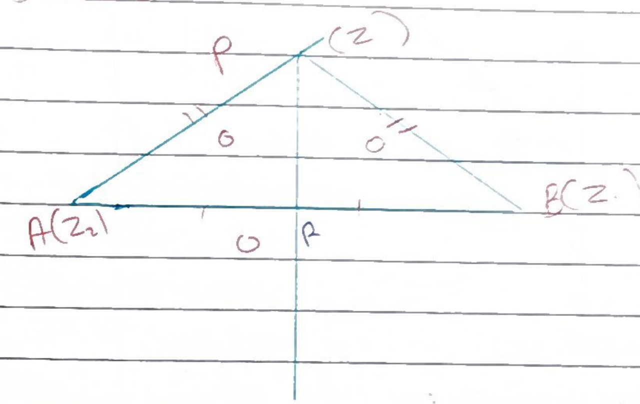
$$\frac{(z_1 - z_0)}{|(z_2 - z_0)|} = \frac{(z_1 - z_0)}{|(z_1 - z_0)|} \cdot e^{i\theta}$$

Geometry of complex Number

Note

I) Use of modulus

1 Equation of Perpendicular bisector



$|\vec{AP}| = |\vec{PB}|$ $\quad$ $|\vec{AR}| = |\vec{BR}|$

$|z - z_1| = |z - z_2|$

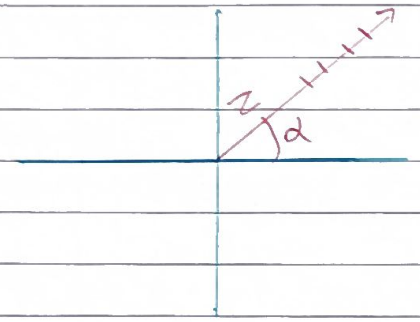
∴ perpendicular bisector

z_1, z_2 are known

Use of argument in Geometry | Note

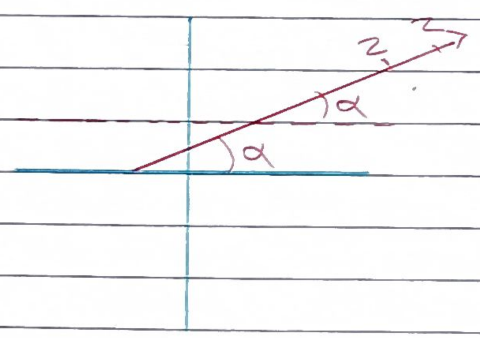
1) Equation of Ray

i) Originating Point is origine



Eq^n : $Arg(z) = \alpha$

ii) Originating Point is not origine



Eq^n : $arg(z - z_0) = \alpha$

De Moivre's theorem

Cas I — If n is an integer

i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i(n\theta)} = (\cos\theta + i\sin\theta)^n$

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$\ldots\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \ldots \theta_n) + i\sin$

$(\theta_2 + \theta_2 + \ldots \theta_n)$

Proof : $e^{i\theta_1} \cdot e^{i\theta_2} \ldots e^{i\theta_n}$

$e^{i(\theta_1 + \theta_2 \ldots \theta_n)}$

$\cos(\theta_1 + \theta_2 \ldots \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots \theta_n)$

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Different form of cube root of unity

Cartesian form	1	$-\frac{1}{2}+\frac{\sqrt{3}i}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}i}{2}$
Polar form	1	$\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$	$\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)$
Euler form	1	$e^{i\left(\frac{2\pi}{3}\right)}$	$e^{-i\left(\frac{2\pi}{3}\right)}$
Notation	1	ω	ω^2
Notation	1	ω	$\overline{\omega}$

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Properties of cube root of unity

1) Sum of cube roots of unity is

$$1+\omega+\omega^2=0$$

2) Product of cube roots of unity is

$$1\cdot\omega\cdot\omega^2=1$$

ω is root $z^3=1$

$$\boxed{\omega^3=1}$$

3) Power of ω

$\omega^3=1$
$\omega^4=\omega$
$\omega^5=\omega^2$
$\omega^6=1$

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☆ n^{th} root of unity

Note

$\boxed{\alpha^n = 1}$

$$z = (1)^{1/n}$$

$$= (\cos 0 + i\sin 0)^{\frac{1}{n}}$$

$$= (\cos(2k\pi) + i\sin(2k\pi))^{\frac{1}{n}}$$

$$z = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = e^{i\left(\frac{2k\pi}{n}\right)}\right)$$

$$\alpha = e = i\left(\frac{2\pi}{n}\right)$$

$k = 0 \quad z = 1 \quad \rightarrow 1$

$k = 1 \quad z = e^{\frac{i2\pi}{n}} \quad \rightarrow \alpha$

$k = 2 \quad z = e^{\frac{i4\pi}{n}} \quad \rightarrow \alpha^2$

$k = 3 \quad z = e^{\frac{i6\pi}{n}} \quad \rightarrow \alpha^3$

$k = n-1 \quad z = e^{i\left(\frac{2(n-1)\pi}{n}\right)} \rightarrow \alpha^{n-1}$

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Properties of n^{th} root of unity

Sum of roots

$$1 + \alpha + \alpha^2 \ldots + \alpha^{n-1} = \boxed{\frac{1(1-\alpha)^n}{1-\alpha}}$$

$$= 0$$

Product of roots is 1 if n is odd & -1 if n is even

$$1 \cdot \alpha \cdot \alpha^2 \ldots \alpha^{n-1} = \alpha^{1+2+\ldots(n-1)}$$

$$= \boxed{\alpha^{\frac{(n-1)n}{2}}}$$

Note: formula

$1+2+3+4+\ldots+n$

$= \frac{n(n+1)}{2}$

$$= \left(e^{\frac{i2\pi}{n}}\right)^{\left(\frac{n-1n}{2}\right)}$$

$$= e^{\frac{i2\pi}{n} \times \frac{(n-1)\times n}{2}}$$

$$= e^{i\pi(n-1)}$$

$$= \cos((n-1)\pi) + i\sin((n-1)\pi)$$

$n = \text{even} \rightarrow -1$

$n = \text{odd} \rightarrow 1$

Note:

$\cos\pi, \cos 3\pi$

$\cos 5\pi = -1$

$\left.\begin{matrix}\cos 2\pi \\ \cos 4\pi \\ \cos 2n\pi\end{matrix}\right] = 1$

Power raised to 'r'

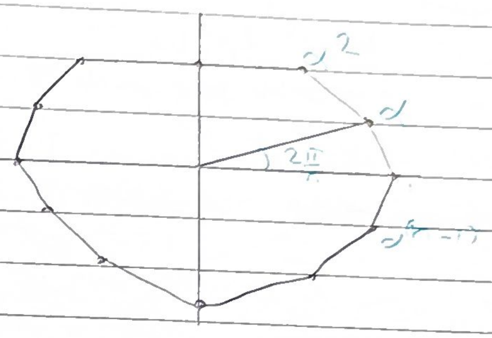
$$1^r + \alpha^r + (\alpha^2)^r + \dots + (\alpha^{(n-1)})^r = \begin{cases} 0, & r \text{ is not multiple of } n \\ n, & r \text{ is multiple of 'n'.} \end{cases}$$

Note :

$1^r + \omega^r + (\omega^2)^r =$ 0, r is not multiple of 3,

3, r is multiple of 3

n^{th} root of unity forms a regular polygon



$$\alpha^n = 1$$

$$\boxed{\alpha \cdot \alpha^{n-1} = 1}$$

$$\alpha^{n-1} = \frac{1}{\alpha} \times \frac{\bar{\alpha}}{\bar{\alpha}} = \frac{\bar{\alpha}}{|\alpha|^2} = \bar{\alpha}$$

$$\boxed{\alpha^{n-2} = \bar{\alpha}^2}$$

Properties of nth root of unity

5) $z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2)\dots(z-\alpha^{n-1})$

$$\frac{z^n-1}{z-1} = (z-\alpha)(z-\alpha^2)\dots(z-\alpha^{n-1})$$

$\downarrow$

$$1 + z + z^2 + \dots + z^{n-1} = (z-\alpha)(z-\alpha^2)\dots(z-\alpha^{(n-1)})$$